Here we have omitted the terms not containing the second derivatives of the generalized coordinates in expression (3.6) and the third derivatives in expression (3.7). From (3.6), (3.7) and (3.4) we obtain the equations of motion in the independent generalized coordinates x and y

$$x^{(2)} + a^2 \frac{x^{\cdot}}{z^{\cdot}} (z^{(2)} + g) = 0, \qquad y^{(2)} + a^2 \frac{y^{\cdot}}{z^{\cdot}} (z^{(2)} + g) = 0$$

They agree with the equations of motion obtained in [8] by another method.

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ON THE STABILITY OF NONLINEAR SYSTEMS WITH A TRANSFORMED ARGUMENT

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We study a linear and a perturbed system; in the latter the argument is transformed. Under the assumption that the trivial solution of the linear system is stable, we ascertain the conditions under which the trivial solution of the perturbed system also will be stable.

Let $f(t, \xi_k) = f(t, \xi_1, \xi_2, \ldots \xi_p)$ $(k = 1, 2, \ldots, p)$, where $f, \xi_1, \xi_2, \ldots, \xi_p$ are m-dimensional vectors. We consider the following two mth-order systems: the linear one

$$y' = A(t)\, y \tag{1}$$

and the perturbed one (see [1])

$$x'(t) = A(t)\,x(t) + f(t,\ x(\varphi_k(t,\ x(t)))) \tag{2}$$

where A is a square matrix and φ_k are transformations of the argument. We study the stability of the trivial solution of system (2) which is to be undestood differently in each of the cases to be considered. We assume that the trivial solution of system (1) is stable. Let us ascertain the conditions under which the trivial solution of system (2) will be stable.

Integrals are everywhere understood in the Lebesgue sense. Measurability, if it is not stipulated, also is understood in the Lebesgue sense. The symbol $\|\ \|$ denotes the norm of a vector or of a matrix, which equals the sum of the absolute values of the elements. By $Y(t)$ we have denoted the matrix which is a solution of system (1), satisfying the initial condition $Y(t_0) = E$, where E is the unit matrix.

Let us assume that the following conditions, which we call conditions ω_1, are fulfilled for system (2). The matrix $A(t)$ is defined, continuous, and bounded for $t \in [t, \infty)$, $\|A(t)\| \leqslant M$; the components of the vector-valued function $f(t, \xi_k)$ are defined and continuous for $t \in [t_0, \infty)$ and $\|\xi_k\| \leqslant R$, where $R > 0$; there holds the inequality

$$\|f(t, \xi_k)\| \leqslant \sum_{k=1}^{p} g_k(t)\,\|\xi_k\| \tag{3}$$

where $g_k(t)$ are continuous functions and

$$\sum_{k=1}^{p} g_k(t) \leqslant N < \infty$$

the functions $\varphi_k(t, \xi)$ are defined for $t \in [t_0, \infty)$ and $\|\xi\| \leqslant R$, satisfy the inequalities $\varphi_k(t, \xi) \leqslant t$, and have continuous partial derivatives in all the variables; there exist numbers $\alpha, c > 0$ such that $\frac{\partial}{\partial t}\varphi_k(t, \xi) \geqslant \alpha, \quad \left|\frac{\partial}{\partial \xi_i}\varphi_k(t, \xi)\right| \leqslant c$

For the given case, as the set of initial vector-valued functions we take a set Z which consists of continuous m-dimensional vector-valued functions, bounded for $t \leqslant t_0$. The trivial solution of system (2) is said to be stable relative to Z if for any $\varepsilon > 0$ there exists $\delta > 0$ such that the solution of system (2), corresponding to any initial vector-valued function $z \in Z$ such that $\|z\| < \delta$, satisfies the inequality $\|x\| < \varepsilon$ for $t \geqslant t_0$. If moreover $\lim x(t) = 0$ as $t \to \infty$, then the trivial solution of system (2) is said to be asymptotically stable relative to Z.

Theorem 1. Assume that

1) conditions ω_1 are fulfilled for system (2);

2) matrix $A(t)$ is periodic or is such that

$$\int_{t_0}^{t} \operatorname{sp} A(\tau)\,d\tau \geqslant \mu > -\infty$$

3) the integrals

$$\int_{t_0}^{\infty} g_k(\tau)\,d\tau$$

converge. Then from the stability of the trivial solution of system (1) follows the stability relative to Z of the trivial solution of system (2).

Proof. Assume that the trivial solution of system (1) is stable. We take a number r

such that the inequalities $\alpha - c(M + N)r > 0$, $0 < r < R$ are fulfilled. We can find a number a (see [2]) such that $\| Y(t)Y^{-1}(\tau) \| \leqslant a$ for $t \in [t, \infty)$, $\tau \in [t_0, t]$. We denote

$$b = \sum_{k=1}^{p} \int_{t_0}^{\infty} g_k(\tau)\, d\tau$$

By virtue of the stability of the trivial solution of system (1) we can find numbers $\delta > 0$ and $\delta_1 > 0$, satisfying the inequality $(\delta + ab\,\delta_1)\exp(ab) < r$, such that if the solution y of system (1) satisfies the inequality $\| y(t_0) \| < \delta_1$ at the initial instant, then $\| y(t) \| < \delta$ for $t \in [t_0, \infty)$. Let us consider a subset Z_1 of set Z, consisting of vector-valued functions satisfying the inequality $\| z \| < \delta_1$. From [3] (Theorem 2.15) follows the existence of a solution of system (2) for any initial vector-valued function $z \in Z_1$. Let us show that each such function is infinitely continuable.

We assume the opposite. Then there exist a solution $x(t)$ and a number $T > t_0$ such that $\| x(T) \| = r$, while $\| x(t) \| < r$ for $t \in [t_0, T)$. For $t \in [t, T]$ we have

$$\frac{d}{dt}\varphi_k(t, x(t)) = \frac{\partial}{\partial t}\varphi_k(t, x(t)) +$$

$$\sum_{i=1}^{m} \frac{\partial}{\partial x_i}\varphi_k(t, x(t))\, x_i'(t) \geqslant \alpha - c\| x'(t) \| \geqslant \alpha - c(M+N)r$$

Therefore, for $t \in [t_0, T]$ the functions $\varphi_k(t, x(t))$ have inverses which we denote by $\Phi_k(t)$, respectively. The functions $\Phi_k(t)$ are defined and continuously differentiable on the intervals $[\varphi_k(t_0, x(t_0)), \varphi_k(T, x(T))]$, respectively. We define these functions in such a way that they are continuously differentiable on the intervals $[\varphi_k(t_0, x(t_0)), T]$ and, that their derivatives are positive.

For a given solution x the equality

$$x(t) = y(t) + \int_{t_0}^{t} Y(t)\, Y^{-1}(\tau)\, f(\tau, x(\varphi_k(\tau, x(\tau))))\, d\tau \tag{4}$$

is valid for $t \in [t_0, T]$, where y is a solution of system (1) with the initial condition $y(t_0) = x(t_0)$; for $t \in [t_0, T]$, we have

$$\| x(t) \| \leqslant \delta + ab\delta_1 + a\sum_{k=1}^{p}\int_{t_0}^{t} g_k(\Phi_k(\tau))\,\Phi_k'(\tau)\,\| x(\tau) \|\, d\tau$$

and, consequently,

$$\| x(t) \| \leqslant (\delta + ab\delta_1)\exp\left(a\sum_{k=1}^{p}\int_{\Phi_k(t_0)}^{\Phi_k(t)} g_k(\tau)\, d\tau\right) \leqslant (\delta + ab\delta_1)\exp(ab)$$

Therefore, $\| x(T) \| < r$. This signifies that the solution $x(t)$ is continuable onto the whole singular semi-interval $[t_0, \infty)$ and that the inequality $\| x(t) \| \leqslant (\delta + ab\,\delta_1)\exp(ab)$ is valid for $t \in [t_0, \infty)$. Therefore, the trivial solution of system (2) is stable. Theorem 1 is proved.

Theorem 2. Assume that

1) conditions ω_1 are fulfilled for system (2);
2) matrix $A(t)$ is periodic;
3) the integrals

$$\int_{t_0}^{\infty} g_k(\tau)\, d\tau$$

converge;

4) the functions $t - \varphi_k(t, \xi)$ are bounded for $t \in [t_0, \infty)$, $\| \xi \| \leqslant R$.

Then from the asymptotic stability of the trivial solution of system (1) follows the asymptotic stability relative to Z of the trivial solution of system (2).

The validity of Theorem 2 follows from the inequality

$$e^{\nu t} \| x(t) \| \leqslant a_1 \exp \left(b_1 \sum_{k=1}^{p} \int_{t_0}^{\infty} g_k(\tau) d\tau \right)$$

where a_1, b_1 and ν are certain positive numbers, which in turn follows from equality (4) if the initial conditions are sufficiently small.

If the transformations φ_k of the argument do not depend upon the solution, then as the set Z we can take the set consisting of vector-valued functions whose components are Borel measurable.

Let us now assume that the following conditions, which we call conditions ω_2, are fulfilled for system (2). The matrix $A(t)$ is defined and bounded for $t \in [t_0, \infty)$ and its elements are measurable on any finite interval $[t_0, T]$: the vector-valued function $f(t, \xi_k)$ is defined for $t \in [t_0, \infty)$ and $\| \xi_k \| \leqslant R$, where $R > 0$, for fixed ξ_k its components are measurable in t on any finite interval $[t_0, T]$, while for fixed t they are continuous in the ξ_k; inequality (3) holds, where $g_k(t)$ are measurable functions on any finite interval $[t_0, T]$ and are bounded for $t \in [t_0, \infty)$; the functions $\varphi_k(t, \xi)$, bounded for $t \in [t_0, \infty)$ and $\| \xi \| \leqslant R$, satisfy the inequalities $\varphi_k \leqslant t$, and for fixed ξ they are measurable in t on any finite interval $[t_0, T]$, while for fixed t they are continuous in ξ; the inequalities $| \varphi_k(t, \xi) - t | \leqslant h_k(t) \| \xi \|$ hold, where $h_k(t)$ are functions which are measurable and integrable on any finite interval $[t_0, T]$.

By Z_λ, where λ is a nonnegative number, we denote the set of m-dimensional vector-valued functions, defined for $t \in (-\infty, t_0]$ and satisfying the following conditions: if $z \in Z_\lambda$ and $t', t'' \in (-\infty, t_0]$, then $\| z(t'') - z(t') \| \leqslant \lambda | t'' - t' |$. For the given case we take the set Z_λ as the set of initial vector-valued functions. The trivial solution of system (2) is said to be stable relative to Z_λ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that the solution of system (2), corresponding to any initial vector-valued function $z \in Z_\lambda$ such that $\| z(t) \| \leqslant R$ and $\| z(t_0) \| < \delta$, satisfies the inequality $\| x \| < \varepsilon$ for $t \geqslant t_0$. If moreover $\lim x(t) = 0$ as $t \to \infty$, the trivial solution of system (2) is said to be asymptotically stable relative to Z_λ.

Theorem 3. Assume that

1) conditions ω_2 are fulfilled for system (2);

2) matrix $A(t)$ is periodic or is such that

$$\int_{t_0}^{t} \operatorname{sp} A(\tau)\, d\tau \geqslant \mu > -\infty$$

3) the integrals

$$\int_{t_0}^{\infty} g_k(\tau)\, d\tau, \qquad \int_{t_0}^{\infty} g_k(\tau)\, h_k(\tau)\, d\tau$$

converge.

Then from the stability of the trivial solution of system (1) follows the stability relative to Z_λ of the trivial solution of system (2) for any $\lambda \geqslant 0$.

To prove Theorem 3 we use equality (4) and direct estimates. The theorem on asymptotic stability can be formulated analogously.

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ON CERTAIN IMPULSE OBSERVATION LAWS

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We consider the problem of optimizing phase coordinate bounds. We obtain the conditions for the solvability of the problem and establish the form of the optimal observation laws. The paper is closely related to [1, 2]. The problem of optimizing the observation process has been studied from another viewpoint in [3, 4].

1. Let a plant's phase coordinate vector $x(t)$ from an n-dimensional Euclidean space R_n be the solution of the system of equations

$$x^{\cdot}(t) = A(t)\,x(t) + b(t), \qquad x(0) = x_0 \qquad (0 \leqslant t \leqslant T) \tag{1.1}$$

The vector $y(t)$ accessible to observation is given by the relations

$$dy(t) = h(t)\,H(t)\,x(t)\,dt + \sigma(t)\,d\xi(t), \quad y(0) = 0 \tag{1.2}$$

The elements of the matrices $A(t)$, $H(t)$, $\sigma(t)$ and $b(t)$ are continuous functions. The random variable $x(0)$ has a Gaussian distribution with the covariance matrix

$$D_0 = M(x_0 - Mx_0)(x_0 - Mx_0)', \; D_0 > 0$$

Here the prime is the sign for transposition, M is the mean, the symbol $D_0 > 0$ signifies the positive definiteness of matrix D_0. The Wiener process $\xi(t)$ does not depend upon $x(0)$, and the matrix $\sigma(t)\,\sigma'(t) > 0$, $0 \leqslant t \leqslant T$. Without loss of generality [2] we can take the dimension of vector $y(t)$ equal to n. The control of the observation process is effected by choice of the scalar function $h(t)$. We consider the linear combination $q'x(T)$ (the nonzero vector $q \in R_n$) is specified). Let $D(T)$ be the covariance matrix of the conditional distribution of vector $x(T)$ under condition $y(s)$, $0 \leqslant s \leqslant T$.

Problem 1. Determine the function $\gamma(t) = h^2(t)$ (the optimal observation law) which minimizes the expression

$$q'D(T)\,q \tag{1.3}$$